

Methods 4 2007

1. Solutions of type $u(r, \theta) = r^\lambda w(\cos \theta)$.

$$\begin{aligned}
 & \frac{\partial}{\partial r} \left[r^2 \sin \theta \lambda r^{\lambda-1} w' \right] - \frac{\partial}{\partial \theta} (\sin \theta r^\lambda w' \sin \theta) \\
 &= \frac{\partial}{\partial r} [r^{\lambda+1} \lambda \sin \theta w'] - \frac{\partial}{\partial \theta} [\sin^2 \theta r^\lambda w'] \\
 &= \lambda(\lambda+1) r^\lambda \sin \theta w - 2 \sin \theta \cos \theta r^\lambda w' + \sin^3 \theta r^\lambda w'' = 0 \\
 \Rightarrow & \lambda(\lambda+1) w - 2 \cos \theta w' + \sin^2 \theta w'' = 0 \\
 \text{(let } z = \cos \theta \Rightarrow (1-z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + \lambda(\lambda+1) w = 0
 \end{aligned}$$

Solutions at $z = \pm 1$ if $\underbrace{\lambda(\lambda+1)}_{\lambda^2 + \lambda} = n(n+1)$

and $w = P_n(z)$.

$$\begin{aligned}
 \lambda^2 + \lambda &= n^2 + n \\
 (\lambda + \frac{1}{2})^2 - \frac{1}{4} &= (n + \frac{1}{2})^2 - \frac{1}{4} \\
 \Rightarrow (\lambda + \frac{1}{2})^2 &= (n + \frac{1}{2})^2 \\
 \Rightarrow \lambda &= \begin{cases} n \\ -(n+1) \end{cases}
 \end{aligned}$$

One solution, regular at $r=0 \Rightarrow$ then $\lambda = n$ and \Rightarrow

$u(r, \theta) = r^n P_n(\cos \theta) \Rightarrow$ general soln is

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \theta)$$

If $n = \lambda = 0$: $(1-z^2)w'' - 2zw' = 0$

satisfied by $P_0(z) = 1$ ✓

$n = \lambda = 1$: $(1-z^2)w'' - 2zw' + 2w = 0$ if $w = z$ ✓

$n = \lambda = 2$: $(1-z^2)w'' - 2zw' + 6w = 0$ if $w = \frac{3z^2-1}{2}$ ✓

$$u(a, \theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos \theta) = \cos^2 \theta$$

∴ i.e. $\cos^2 \theta = A_0 + A_1 a \cos \theta + A_2 a^2 \left[\frac{3}{2} \cos^2 \theta + \frac{1}{2} \right] + \dots$

$\Rightarrow A_1 = A_3 = \dots = 0.$

$$A_0 = \frac{A_2 a^2}{2} \quad \text{and} \quad A_2 = \frac{2}{3a^2} \Rightarrow A_0 = \frac{1}{3}.$$

≡

$$1. \quad 4x^2y'' + 2x^2y' + (1-x)y = 0$$

also can be written $y'' + \frac{1}{2}y' + \frac{1-x}{4x^2}y = 0$

then $p(x) = \frac{1}{2}$
 $q(x) = \frac{1-x}{4x^2}$ both singular
 but $x^2q(x) = \frac{1-x}{4}$, analytic } R.S.P.

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+\lambda}$$

$$\rightarrow 4x^2 \sum (k+\lambda)(k+\lambda-1)a_k x^{k+\lambda-2} \\ + 2x^2 \sum (k+\lambda)a_k x^{k+\lambda-1} \\ + (1-x) \sum a_k x^{k+\lambda}$$

$$\rightarrow \sum 4(k+\lambda)(k+\lambda-1)a_k x^{k+\lambda} \\ + \sum 2(k+\lambda)a_k x^{k+\lambda+1} \\ + \sum a_k x^{k+\lambda} \\ - \sum a_k x^{k+\lambda+1} \quad \left. \right\} \Rightarrow \begin{aligned} & 4(k+\lambda)(k+\lambda-1)a_k \\ & + 2(k-1+\lambda)a_{k-1} \\ & + a_k \\ & - a_{k-1} \end{aligned} = 0 \\ \rightarrow [4(k+\lambda)(k+\lambda-1) + 1]a_k + [2(k-1+\lambda)-1]a_{k-1} = 0$$

initial eqn: $k=0 \Rightarrow 4\lambda(\lambda-1)+1=0$
 $4\lambda^2-4\lambda+1=0$
 $(2\lambda-1)^2=0 \rightarrow \lambda=\frac{1}{2}$, twice.

rec. rel: $a_k = -\frac{2(k-1+\lambda)-1}{4(k+\lambda)(k+\lambda-1)+1} a_{k-1} = \frac{-2k+3-2\lambda}{4(k+\lambda)(k+\lambda-1)+1} a_{k-1}$

$$a_1 = \frac{-2+3-2\lambda}{4(\lambda+1)\lambda+1} = \frac{1-2\lambda}{4\lambda^2+4\lambda+1} = \frac{1-2\lambda}{(2\lambda+1)^2}$$

$$a_2 = \frac{-4+3-2\lambda}{4(2+\lambda)(\lambda+1)+1} = \frac{(-2\lambda-1)}{4(2\lambda+2+\lambda^2+\lambda)+1} a_1 = \frac{(-2\lambda-1)a_1}{4\lambda^2+12\lambda+9} = \frac{-(2\lambda+1)a_1}{(2\lambda+3)^2}$$

$$\Rightarrow a_k = \frac{(-1)^k (\lambda - \frac{1}{2})}{2^k (k+\lambda-\frac{1}{2})^2 (k+\lambda-\frac{3}{2}) \cdots (\lambda+\frac{3}{2})(\lambda+\frac{1}{2})}$$

$$y_2(x) = \frac{d}{dx} y_1(x). \quad \text{Use log}$$

$$= \frac{d}{dx} \left[\sum a_k x^{k+\lambda} \right] = \sum \left[\frac{da_k}{dx} x^{k+\lambda} + \ln x \cdot a_k \cdot x^{k+\lambda} \right]$$

$$\frac{d}{dx} (\ln a_k) = \frac{1}{a_k} \frac{da_k}{dx}.$$

$$\begin{aligned} \ln a_k &= \ln(-1)^k + \ln(\lambda - \frac{1}{2}) - \ln(2^k) - \ln(k + \lambda - \frac{1}{2}) \\ &\quad - \sum_{n=0}^{k-1} \ln(\lambda + \frac{1}{2} + n) \end{aligned}$$

$$\rightarrow \frac{1}{a_k} \frac{da_k}{dx} = \frac{1}{\lambda - \frac{1}{2}} - \frac{1}{k + \lambda - \frac{1}{2}} - \sum_{n=0}^{k-1} \frac{1}{\lambda + \frac{1}{2} + n}.$$

[Eval'd at $\lambda = \frac{1}{2}$], $a_k \rightarrow 0$.

$$\frac{da_k}{dx} = \dots - \dots$$

3. (a) $\sum_{n=-\infty}^{\infty} t^n J_n(x) = e^{\frac{1}{2}x(t-\frac{1}{t})}$

(i) differentiate $\frac{dx}{dt} \sum t^n J_n(x) = \frac{1}{2}(t - \frac{1}{t}) e^{\frac{1}{2}x(t-\frac{1}{t})}$

also $\sum t^{n-1} J_{n-1}(x) = \frac{1}{t} e^{\frac{1}{2}x(t-\frac{1}{t})}$
 $\sum t^{n+1} J_{n+1}(x) = \frac{1}{t} e^{\frac{1}{2}x(t-\frac{1}{t})}$

$\Rightarrow 2 \sum t^n J_n'(x) = \sum t^{n+1} J_{n+1} - \sum t^{n-1} J_{n-1}$

also $\sum t^n J_{n-1}(x) = t e^{\frac{1}{2}x(t-\frac{1}{t})}$

and $\sum t^n J_{n+1}(x) = \frac{1}{t} e^{\frac{1}{2}x(t-\frac{1}{t})}$

$\Rightarrow 2 \sum t^n J_n'(x) = \sum t^n J_{n-1}(x) - \sum t^n J_{n+1}$

$\Rightarrow 2 J_n'(x) = \underbrace{J_{n-1}(x) - J_{n+1}(x)}$

(ii) differentiate $\frac{dt}{dx} \sum n t^{n-1} J_n(x) = \frac{1}{2} x e^{\cancel{\frac{1}{2}x(t-\frac{1}{t})}} \cdot (1 + \frac{1}{t^2})$

i.e. $\sum n t^{n-1} J_n(x) = \frac{1}{2} x e^{\frac{1}{2}x(t-\frac{1}{t})} (1 + \frac{1}{t^2})$

i.e. $\sum \frac{2n}{x} t^{n-1} J_n(x) = \sum t^n J_{n-1} + \sum t^{n-1} J_{n+1}$

$\Rightarrow 2n J_n(x)/x = \underbrace{J_{n-1}(x) + J_{n+1}(x)}$

(iii). $t = e^{i\theta}$: $\sum e^{i\theta n} J_n(x) = e^{\frac{1}{2}x(e^{i\theta} - e^{-i\theta})}$
 $\sum e^{i\theta n} J_n(x) = e^{x i \sin \theta} = \cos(x \sin \theta) + i \sin(x \sin \theta)$

integrate $\int_0^{2\pi} \sum e^{i\theta n} J_n(x) d\theta = \int_0^{2\pi} \cos(x \sin \theta) + i \sin(x \sin \theta) d\theta$

$\Rightarrow 2\pi J_0(x) = \int_0^{2\pi} \cos(x \sin \theta) d\theta$

$$(b) \quad x^2 y'' + xy' + p^2 x^2 y = 0$$

$$\frac{d}{dx} \left[x^2 \left(\frac{dy}{dx} \right)^2 \right] + p^2 x^2 \frac{d}{dx} (y^2) = 0$$

$$\frac{d}{dx} [2yy'] = 2y''$$

$$\begin{aligned}
 &= 2x \left(\frac{dy}{dx} \right)^2 + 2x^2 \frac{d^2 y}{dx^2} + 2p^2 x \frac{d}{dx} (y^2) + p^2 x^2 \frac{d^2}{dx^2} (y^2) = 0 \\
 &= 2x(y')^2 + 2x^2 y'' y' + 4p^2 x y y' + p^2 x^2 (2(y')^2 + 2y y'') \\
 &= 2x(y')^2 + 2x^2 y'' y' + 4p^2 x y y' + p^2 x^2 2(y')^2 + 2p^2 x^2 y y'' \\
 &= 2x(y')^2 + x^2 2y' y'' + p^2 x^2 2y y' \\
 &= 2y' [2x y' + x^2 y'' + p^2 x^2 y] = 0
 \end{aligned}$$

Presumably need to use: $\frac{d}{dx} \left[x^2 \left(\frac{dy}{dx} \right)^2 \right] + p^2 x^2 \frac{d}{dx} (y^2) = 0$

$y = J_0(\rho x)$ satisfies this.

$$\begin{aligned}
 &\hookrightarrow \frac{d}{dx} \left[x^2 [J'_0(\rho x)]^2 \right] + p^2 x^2 2 J_0(\rho x) J'_0(\rho x) \\
 &\rightarrow \left[x^2 [J'_0(\rho x)]^2 \right]_0^l = - \int_0^l p^2 x^2 2 J_0(\rho x) J'_0(\rho x) \\
 &\rightarrow \left[l^2 [J'_0(\rho l)]^2 \right] = - \int_0^l p^2 x^2 \cdot \frac{d}{dx} [(J_0(\rho x))^2] \\
 &= -p^2 \left[x^2 [J'_0(\rho x)]^2 \right]_0^l + \int_0^l 2x [J_0(\rho x)]^2 dx \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 t. (a) \quad \widehat{f * g} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) dy e^{-ikx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-ik(x-y)} e^{-iky} dy dy dx \\
 &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} g(y) e^{-iky} dy \cdot \int_{-\infty}^{\infty} f(x-y) e^{-ik(x-y)} dx \right) \\
 u = x-y &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{-iky} dy \cdot \int_{-\infty}^{\infty} f(u) e^{-iku} du \\
 &= \sqrt{2\pi} \widehat{f} \widehat{g}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow f * g &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi} \widehat{f} \widehat{g} e^{ikx} dx \\
 &= \int_{-\infty}^{\infty} \widehat{f} \widehat{g} e^{ikx} dx
 \end{aligned}$$

$$\text{at } x=0 \Rightarrow \int_{-\infty}^{\infty} \widehat{f}(k) \widehat{g}(k) dk = \int_{-\infty}^{\infty} f(-y) g(y) dy ,$$

$$(b) h(x) = \frac{1}{x^2 + a^2} \quad a > 0$$

$$\begin{aligned}
 \Rightarrow \widehat{h}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2 + a^2} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{(x+ai)(x-ai)} \\
 &= \frac{1}{\sqrt{2\pi}} \cdot \cancel{2\pi i} \operatorname{Res} \left(\frac{e^{-ikx}}{(x+ai)(x-ai)}, ai \right) \quad \text{poles at } x=ai, -ai \\
 &\quad \cancel{2\pi i} \\
 &= \sqrt{\frac{\pi}{2}} \cdot \cancel{\frac{1}{2\pi i}} \frac{e^{-ik\cdot ai}}{2ai} = \sqrt{\frac{\pi}{2}} \frac{e^{ka}}{a}
 \end{aligned}$$

If I do $\boxed{\int}$, I'll show it's $\underbrace{\sqrt{\frac{\pi}{2}} \frac{e^{ika}}{a}}$.

So using convolution theorem with $f=g=h$,

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} &= \widehat{f * f} = \sqrt{2\pi} \widehat{h} = \sqrt{2\pi} \frac{\pi}{2} \frac{e^{-2aka}}{a^2} \\
 \int_{-\infty}^{\infty} \frac{\pi}{2} \frac{e^{-2aka}}{a^2} dk &= \frac{2\pi}{2a^2} \int_0^{\infty} e^{-2ak} dk = \frac{\pi}{2a^3}
 \end{aligned}$$

$$5. \phi_{xx} + \phi_{yy} = 0$$

What is $\hat{\phi}_{yy}$? - $\Rightarrow \hat{\phi}'' - k^2 \hat{\phi} = 0$
 $\Rightarrow \hat{\phi} = A e^{-iky}$

on $y=0$, $\phi = \begin{cases} \frac{1}{2} & |x|<1 \\ 0 & |x|>1 \end{cases}$

$$\Rightarrow \hat{\phi} = A = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \begin{cases} \frac{1}{2} & |x|<1 \\ 0 & |x|>1 \end{cases} \right\} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_0^1 \cos kx dx = \frac{1}{\sqrt{2\pi}} \frac{\sin k}{k}$$

$$\Rightarrow \hat{\phi} = \frac{1}{\sqrt{2\pi}} \frac{\sin k}{k} e^{-iky}$$

C Inverky $\Rightarrow \phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \frac{\sin k}{k} e^{-iky} dk$
 $= \frac{1}{\pi} \int_0^{\infty} \frac{\cos kx \sin k}{k} e^{ky} dk$

$$\text{at } x = \pm 1 \text{ and } y = 0, \Rightarrow = \frac{1}{\pi} \int_0^{\infty} \frac{\sin 2k}{2k} dk = \frac{1}{\pi} \int_0^{\infty} \frac{\sin u}{u} \frac{du}{2} = \underline{\underline{\frac{1}{4}}}.$$

expected as discontinuous

6. γ is chosen to be to the right of all singularities.

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{ts - a\sqrt{s}} ds$$